In the case of series (5) which has been obtained, it is characteristic that each homogeneous solution is represented in the form of a sum for any instant of time t.

We now present the results of numerical calculations in the case when

$$\varepsilon = 1,\ f(\zeta, t) = \varphi(\zeta)\, g(t),\ \varphi(\zeta) = \zeta^2,\ g(t) = H(t) - H(t - 0.2)$$

where $H(t)$ is the Heaviside function. The time a/c is adopted for T_0 where $c = \sqrt{\mu/\rho}$ is the velocity of propagation of shear waves.

The stress distribution $\tau_{\xi y}$ in time at the point $\xi = \zeta = 0.8$ (Fig.1) reflects the transient nature of the change in the stressed state. In the interval $t \in [0.2, 0.4]$, the behaviour of the stresses $\tau_{\xi y}$ at this point is only slightly different from the behaviour of the function $g(t)$. However, during the interval when the perturbations are reflected from the boundaries $\xi = \pm 1$, the stresses change rapidly both in magnitude and in sign. During the intervals of time after the passage of the rear front of the wave as time increases, the oscillation of the stresses increases and changes sign.

The distribution of the stresses with respect to ξ at the instant of time $t = 2.5$ when $\zeta = 0, 0.5, 1$ (curves *1*, *2* and *3*) is shown in Fig.2. When $\zeta = 0$, if the "jumps" in the stresses when $\xi = 0.5$ and $\xi = 0.7$ are insignificant in magnitude, then, as ζ increases, they become larger and are accompanied by a change in sign.

When $t = 2.5$, curves *1-3* in Fig.3 reflect the stress distribution with respect to ζ when $\xi = 0.2$ (curve *1* $(10\tau_{\xi y})$, $\xi = 0.5$ (curve *2*, $(10\tau_{\xi y})$, and $\xi = 0.8$ (curve *3*).

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AN APPROACH TO SOLVING THE PROBLEM OF A CRACK IN A WEDGE-SHAPED PART OF A PLANE*

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In a development of previous obtained results of the solution of a problem concerning a crack which emerges orthogonally onto the boundary of a half-plane /1/, the problem of a crack of finite length on the axis of symmetry of one of the wedge-shaped parts of a plane is considered. The indices of the singularities of the solution are determined at both vertices of the crack and expressions are presented for the coefficients accompanying these singularities. Numerical values of the coefficients of the stress intensity are obtained in the case when the parts are opened at a right angle and there is a constant load on the edge of the crack. These results are in agreement with data cited in the literature for a piecewise homogeneous plane with a slit which emerges orthogonally onto the line where the half-planes join /2/.

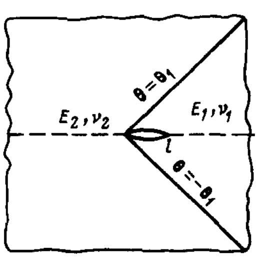

Fig.1

Previously /3/, a solution of the functional Wiener-Hopf equation was presented in closed form for an analogous problem and an expression was given for the coefficient accompanying the fractional power singularity of the solution, that is, at the right end of the slit. Most attention will therefore be paid to isolating the singularities of the two vertices of the crack and to determining the coefficients accompanying these singularities.

1. Formulation of the problem. Reduction to the Riemann problem.

A crack of finite length is considered which emerges along the axis of symmetry of one of the wedge-shaped parts of a piecewise homogeneous plane onto the line where the materials are joined (Fig. 1). A selfbalanced load $\sigma_\theta(r, 0) = -f(r)$ is applied to the edges of

*Prikl.Matem.Mekhan., 52, 4, 699-703, 1988 the crack. The derivative of the displacements in the circumferential direction has a discontinuity on passing across the crack

$$\frac{\partial}{\partial r}[v(r,-0)-v(r,+0)] = -\frac{\varkappa+1}{2\mu}\varphi(r) \tag{1.1}$$
$$\varkappa = \frac{3-\nu}{1+\nu}, \quad \mu = \frac{E}{2(1+\nu)}$$

Here ν is Poisson's ratio, E is Young's modulus and $\varphi(r)$ is an unknown function which has singularities as $r\to 0$ and $r\to 1$ (a crack with a reduced length $l=1$ is considered without any loss of generality).

In accordance with the boundary conditions (1.1) and the symmetry of the problem about the line $\theta=0,\ \theta=\pi,\ r>0$, we write the condition for the undeformed character of the line $\{\theta=\pi,\ r>0;\ \theta=0,\ r>1\}$: $\partial v(r,0)/\partial r=0$ and, when $\theta=0,\ 0<r<1$, we have $\partial v(r,+0)/\partial r = -(4\mu)^{-1}(\varkappa+1)\varphi(r)$.

The rest of the formulation of this problem is identical to that presented in /3/:

$$\langle\sigma_\theta\rangle = \langle\tau_{r\theta}\rangle = \langle u\rangle = \langle v\rangle = 0 \ (\theta=\theta_1);$$
$$\tau_{r\theta}(r,\theta) = 0\ (\theta=0,\pi);\ \sigma_\theta = -f(r)$$

when $\theta=0,\ 0<r<1$. The angle brackets denote a discontinuity in the values of the corresponding components of the displacements and stresses on crossing the line where the materials are joined.

Taking account of the well-known* (*Tikhonenko L.Ya., Boundary value problems for partial differential equations in wedge-shaped domains leading to the Karleman problem, Candidate Dissertation, Odessa Gos. Univ., Odessa, 1975.) representations of elastic displacements and stresses in terms of Mellin integrals for problems with wedge-shaped domains (the integration is carried out along the straight line parallel to the imaginary axis)

$$2\mu v = \frac{1}{2\pi i}\int [p_2(A_0s_2 - B_0c_2) + (\varkappa+p)(-A_1s_1 + B_1c_1)]\, r^{-p}\, dp$$
$$2\mu u = \frac{1}{2\pi i}\int [p_2(A_0c_2 + B_0s_2) + (\varkappa+p)(A_1c_1 + B_1s_1)]\, r^{-p}\, dp$$
$$\sigma_\theta = \frac{1}{2\pi i}\int pp_2(A_0c_2 + B_0s_2 + A_1c_1 + B_1s_1)\, r^{-p_1}\, dp$$
$$s_i = \sin p_i\theta,\ c_i = \cos p_i\theta,\ p_1 = p+1,\ p_2 = p-1$$

we arrive at the Riemann problem by solving two systems of algebraic equations with respect to four unknown functions ($\Psi_+(r)$ is an unknown integrable function):

$$\Phi_-(p_0)\,G(p_0) = 4\pi F_-(p_0) + \Psi_+(p_0) \tag{1.2}$$
$$-1 < \operatorname{Re} p_0 < 0, \quad \Phi_-(p) = \int_0^1 \varphi(r)\, r^p\, dr$$
$$F_-(p) = \int_0^1 f(r)\, r^p\, dr, \quad \Psi_+(p) = \int_1^\infty \Psi_+(r)\, r^p\, dr$$
$$\sigma_\theta(r,0) = \begin{cases} -f(r), & 0<r<1 \\ -\Psi_+(r), & r>1 \end{cases}$$

$$G(p) = {}^1/_2\pi R_2^{-1}[(p_1\omega_0^4 + \omega_0^2)(p_2\omega_1^3 - \omega_1^1) - (p_2\omega_0^3 + \omega_0^1)(p_1\omega_1^4 - \omega_1^3)]$$
$$\omega_0^1 = p_2\nabla_1 - \Delta_4 p_2\rho(2p_2),\ \omega_1^1 = \lambda[\Delta_2\rho(2) + \Delta_1\rho(2\rho)],\ \omega_0^2 =$$
$$p_2[\operatorname{tg}\pi p\nabla_1 + \Delta_4\rho'(2p_2)],\ \omega_1^2 = \lambda[-\Delta_1\rho'(2p) + \Delta_2\rho'(2)]$$
$$\omega_0^3 = \Delta_3\rho(2p),\ \omega_1^3 = \nabla_2 + \Delta_4\rho(2p_1)$$
$$\omega_0^4 = -\Delta_3\rho'(2p),\ \omega_1^4 = \nabla_2\ \operatorname{tg}\pi p - \Delta_4\rho'(2p_1)$$
$$\rho(x) = \cos x\theta_1 + \sin x\theta_1 \operatorname{tg}\pi p,\ \rho'(x) = \theta_1^{-1} d\rho/dx$$
$$\Delta_1 = p^2k_1 - k_2,\ \Delta_2 = p(k_2 - k_1),\ \Delta_3 = \lambda p_2 k_1$$
$$\Delta_4 = \lambda p k_1,\ \nabla_i = 1 + \lambda k_i,\ i = 1, 2$$
$$k_1 = K - 1,\ k_2 = \varkappa_1 K - \varkappa_2,\ K = \mu_1/\mu_2$$
$$\lambda = (\varkappa_2 + 1)^{-1},\ R_2 = \omega_0^2\omega_1^4 + \omega_0^4\omega_1^2$$

The coefficient for the problem for $\theta_1=\pi/2$ is identical to that cited in /2/. It should be noted that the functional Eq.(3) was written in /3/ in the domain $-1<\operatorname{Re} p<1$ and the strip was defined by the inequalities $-\varepsilon_1<\operatorname{Re} p<\varepsilon_2$ where $0<\varepsilon_i<1,\ i=1,2$ and it was not indicated how ε_i were chosen. However, by taking account of the singularity in the stress field $\sigma\in(-1,0)$ at the left end of the cut and the behaviour at infinity, it is necessary to indicate the strip of regularity of the function $\Phi_-(p)$ in a somewhat refined form: $-1<\operatorname{Re} p<0$.

2. Isolation of the singularities at the two vertices of the cut and

expressions for the coefficients accompanying these singularities. Let us shift the strip of regularity of the function $\Phi_-(p_0)$: $0 < \operatorname{Re} p_0 < 1$ by making the substitution $p_0 = p_0 + 1$ By writing $G(p_0)$ in the form

$$G(p_0) = \pi V(p_0)/\operatorname{tg} \pi p_0 \tag{2.1}$$

it can be shown that $V(p_0) \to 1$ $(|p_0| \to \infty)$, $[\arg V(p_0)]_{\sigma-i\infty}^{\sigma+i\infty} = 0$, $\max(\alpha, \sigma_0, {}^1/_2) < \sigma < 1$ where $G(\sigma_0) = 0$, $\sigma_0 \in R$, $0 < \sigma_0 < 1$ and $f(r) = o(r^{-\alpha})$ and, moreover, it is possible that $0 < \alpha < 1$. In this case, factorization

$$V(p_0) = X^+(p_0)/X^-(p_0) \tag{2.2}$$

$$X(p) = \exp\left\{\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\ln V(s)}{s-p}\, ds\right\}$$

taking account of the representation

$$\operatorname{tg} \pi p = A^+(p)\, A^-(p)\, D(p) \tag{2.3}$$

$$A^+(p) = \frac{\Gamma({}^3/_2 - p)}{\Gamma(2-p)}, \quad A^-(p) = \frac{\Gamma({}^1/_2 + p)}{\Gamma(p)}$$

$$D(p) = \frac{1-p}{{}^1/_2 - p}$$

(here, $\Gamma(p)$ is a gamma-function) enables one to rewrite condition (1.2) in the form

$$\frac{\Phi_-(p_0)}{D(p_0)\, A^-(p_0)\, X^-(p_0)} = \frac{4F_-(p_0) A^+(p_0)}{X^+(p_0)} + \frac{\Psi_+(p_0)\, A^+(p_0)}{\pi X^+(p_0)}$$

The subsequent manipulations are analogous to those carried out in /1/ and hence we immediately write out the solution of the Riemann problem:

$$\Phi_-(p) = \left[\frac{c}{p-1} - 4Q_-(p)\right] X^-(p)\, A^-(p)\, D(p) \tag{2.4}$$

$$Q(p) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{F_-(s)\, A^+(s)}{X^+(s)}\, \frac{ds}{s-p}$$

Taking account of the representation of the solution (2.4) at infinity, we write the unknown derivative of the discontinuity in the displacements at the right end of the cut in the form

$$\varphi(r) = \frac{A^\circ}{\sqrt{-\pi \ln r}} + A^1 \frac{(-\ln r)^{\beta-1}}{\Gamma(\beta)}, \quad r \to 1$$

$$A^\circ = c + 4\eta A^+(-\gamma)/X^+(-\gamma), \; f(r) = \eta r^\gamma$$

$$\beta = {}^1/_2 + \varepsilon, \; \varepsilon \to 0, \; \varepsilon > 0$$

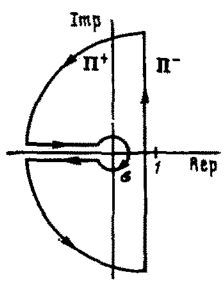

Fig.2

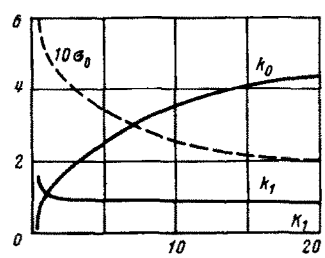

Fig.3

where c is a constant which is determined from the condition of the closedness of the crack

$$\int\limits_0^1 \varphi(r)\, dr = 0$$

The original function

$$\varphi(r) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \Phi_-(p_0)\, r^{-p_0}\, dp_0$$

is determined by closure of the integration contour in the Π^+ half-plane (Fig.2) which is cut along the ray $\operatorname{Re} p \leqslant 0$, $\operatorname{Im} p = 0$.

We note that it was not possible in /3/ to obtain such a representation by applying an inverse Mellin transform to $\Phi_-(p)$.

At the vertex of the cut, which emerges onto the line where the parts of the plane are joined, we obtain, when expressions (2.1)-(2.3) are taken into account,

$$\varphi(r) = Br^{\zeta} + o(r^{\zeta}),\ \zeta = -\max(\alpha, \sigma_0),\ r \to 0$$
$$B = 4\pi/G(-\gamma)\ (\zeta = -\alpha)$$
$$B = \operatorname{res}\left[\frac{\pi}{G(p)}, p = \sigma_0\right]\frac{X^+(\sigma_0)}{A^+(\sigma_0)}\left[\frac{c}{1-\sigma_0} - \frac{4\eta A^+(-\gamma)}{(\gamma+\sigma_0)X^+(-\gamma)}\right]\ (\zeta = -\sigma_0)$$

When account is taken of the singularity in $\varphi(r)$ as $r \to 0$, this enables one to establish the dependence of the displacement field and the stress field on the different combinations of materials. The singularity in the derivative of the displacement discontinuity at the left end of the cut was not isolated in /3/ and no expression was given for the coefficient accompanying this singularity.

3. Numerical analysis of the solution. In the case when $\theta_1 = \pi/2$, we arrive at the problem of a crack which emerges orthogonally onto the line joining the half-planes. In accordance with /2/, the coefficients of the stress intensity take the form

$$k_0 = K_1\frac{(2\sigma_0 - 1)d_1 + (3 - 2\sigma)d_2}{d_1 d_2 \sin\pi\sigma_0}B, \quad K_1 = \frac{E_2}{E_1}$$
$$k_1 = -\frac{1}{2\sqrt{\pi}}\left[c + \frac{4\eta A^+(-\gamma)}{X^+(-\gamma)}\right]$$
$$d_1 = K_1(1+\nu_2) + 3 - \nu_1,\ d_2 = K_1(3-\nu_2) + 1 - \nu_1$$

(k_0 is for the left vertex and k_1 for the right vertex).

The values of k_0 and k_1 (the solid lines) were obtained for different combinations of materials (epoxide - aluminium, aluminium - steel, etc.) taking account of the effect of the latter on the index of the singularity in the solution at the left end of the cut (Fig.3, the broken line). The results which have been presented correspond to the data obtained in /2/ by approximate methods.

Hence, the technique described in /1/ for solving the problem of a crack which emerges orthogonally onto the boundary of a half-plane enables one to obtain exact solutions for a number of problems concerned with cracks. As an analysis of /4/ shows, an identical approach to the problem of a semi-infinite stringer on the axis of symmetry of a wedge - shaped part of a plane also leads to an exact solution of the functional equation.

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